1. Introduction - Asymptotic approach to system reliability

In the reliability investigation of large-scale systems, the problem of the complexity of their reliability functions arises. This problem may be approximately solved by assuming that the number of system components tends to infinity and finding the limit reliability function of the system. This approach is well recognized for basic systems. Gnedenko [1] has solved it for series and parallel systems, whereas Smirnov [7] for “k out of n” systems. They both have found the classes of possible limit reliability functions of these systems. All current results on asymptotic approach to reliability of large systems with typical structures are partly given in [2] and [4]-[6] and completely presented in the monograph [3].

In the paper these two areas considered by Gnedenko and Smirnov are brought together and some new results of investigations are showed. We assume that the lifetime distributions do not necessarily have to be concentrated on the interval <0,∞). Then, a reliability function does not have to satisfy the usually demanded condition

\[ R(t) = 1 \] for \( t \in (-\infty, 0) \).

It is a generalization of the usually used concept of a reliability function. This generalization is convenient in the theoretical considerations. At the same time, the achieved results for the generalized reliability functions, also hold for the usually used reliability function. From these agreement it follows that between a reliability function \( R(t) \) and a distribution function \( F(t) \) there exists an explicit correspondence given by

\[ R(t) = 1 - F(t), t \in (-\infty, \infty). \]

According to the properties of a distribution function, a reliability function \( R(t) \) is non-increasing, right-continuous, \( R(-\infty) = 1 \) and \( R(\infty) = 0 \).

We will deal with a reliability functions of the forms:

\[ R^{(m)}(t) = 1 - \sum_{i=0}^{m-1} \exp[-V(t)] \frac{[V(t)]^i}{i!}, \quad (1) \]

\[ R^{(4)}(t) = 1 - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{-L(t)} \exp[-\frac{x^2}{2}] dx, \quad (2) \]

and

Examples of series-“m out of k” systems and their limit reliability functions

Keywords
reliability, large system, asymptotic approach, limit reliability function

Abstract
The paper is concerned with mathematical methods in asymptotic approach to systems reliability analysis. The complexity of the reliability investigation of large-scale systems is proposed to be approximately solved by assuming that the number of system components tends to infinity and finding the limit reliability function of the system. Some general results in the form of auxiliary theorems and examples of limit reliability functions of homogeneous and regular series-“m out of k” systems with exponential and Weibull reliability functions of system components are presented.
\[
\mathcal{R}(t) = \sum_{i=0}^{\infty} \exp[-(t+T_i)] \frac{[\mathcal{V}(t)]^i}{i!},
\]
where \( m \in N, \lambda \in (0,1) \) and \( \overline{m} \in N. \)

The Asymptotic approach to systems reliability is based on investigating limit distributions of a standardized random variable

\[
(T - b_n)/a_n,
\]
where \( T \) is the lifetime of the system and \( a_n > 0, b_n \in (-\infty, \infty) \) are some suitably chosen numbers. And since

\[
P\left(\frac{T - b_n}{a_n} > t\right) = P(T > a_n t + b_n) = R_n(a_n t + b_n),
\]
where \( R_n(t) \) is a reliability function of the system, then we assume the following definition.

**Definition 1.** A reliability function \( \mathcal{R}(t) \) is called the limit reliability function of the system if there exist normalising constants \( a_n > 0, b_n \in (-\infty, \infty) \) such that

\[
\lim_{n \rightarrow \infty} R_n(a_n t + b_n) = \mathcal{R}(t) \text{ for } t \in C_{\mathcal{R}},
\]
where \( C_{\mathcal{R}} \) is the set of continuity points of \( \mathcal{R}. \)

Hence, for sufficiently large \( n \), we get the following approximate formula approximate formula

\[
R_n(t) \equiv \mathcal{R}(t - b_n/a_n), \ t \in (-\infty, \infty).
\]

### 2. Reliability of series-“m out of k” systems

Suppose that

\[
E_{ij}, i = 1, 2, \ldots, k_n, \ j = 1, 2, \ldots, l_n, \ k_n, l_n \in N
\]
are components of a system having reliability functions

\[
R_{ij}(t) = P(T_{ij} > t), \ t \in (-\infty, \infty),
\]
where \( T_{ij} \) are independent random variables representing the lifetimes of \( E_{ij} \), having distribution functions

\[
F_{ij}(t) = P(T_{ij} \leq t), \ t \in (-\infty, \infty).
\]

**Definition 2.** A system is called regular series-“m out of k” if its lifetime is given by

\[
T = T_{(k_n - m_n + 1)}, \ 0 < m_n \leq k_n,
\]
where \( T_{(k_n - m_n + 1)} \) is the \( m_n \)-th maximal order statistics in a sample of random variables

\[
T_i = \min\{T_{1j}, \ldots, T_{ij}, \ldots, T_{lj}\}, \ i = 1, 2, \ldots, k_n,
\]
representing the lifetimes of series subsystems of the system.

The above definition means that a series-"m out of k_n” system is not failed if and only if at least \( m_n \) of its \( k_n \) series subsystems are not failed.

**Definition 3.** A regular series-“m out of k_n” system is called homogeneous if component lifetimes \( T \) are independent random variables \( T \) representing the lifetimes of series subsystems of a system having distribution function

\[
R(t) = 1 - F(t), \ t \in (-\infty, \infty).
\]

The reliability function of the homogeneous regular series-“m out of k_n” system is given by

\[
R_{k_n, l_n}^{(m_n)}(t) = 1 - \sum_{i=0}^{m_n-1} \binom{k_n}{i} [R^l_n(t)]^i [1 - R^l_n(t)]^{k_n-i},
\]
or by formula

\[
\overline{R}_{k_n, l_n}^{(m_n)}(t) = \sum_{i=0}^{\overline{m}} \binom{k_n}{i} [1 - R^l_n(t)]^i [R^l_n(t)]^{k_n-i},
\]
where \( t \in (-\infty, \infty), \overline{m} = k_n - m_n, k_n \) is the number of series subsystems of a system and \( l_n \) is the number of components in series subsystems.

### 3. Examples of series-“m out of k_n” systems and their limit reliability functions

It is important to notice, that the form of limit reliability function of homogeneous regular series-“m out of k_n” system depends not only on the reliability function of system components, but also on relation between \( m_n \) and the number \( k_n \) of series subsystems of the system and moreover between \( k_n \) and the number \( l_n \) of components in series subsystems of our system. The paper presents some spectacular solutions for the problem of possible limit reliability functions for homogeneous and regular series-“m out of k_n” with
Weibull or exponential reliability function of the system components. The proofs of presented lemmas can be found in [2] and [5].

Agreement 1. We assume the following notation for any positive functions $x(n)$ and $y(n)$:

$$y(n) = o(x(n)) \text{ means that } \lim_{n \to \infty} \frac{y(n)}{x(n)} = 0.$$ 

If $x(n) = o(1)$ and numbers $a, \alpha$ are such that $a \neq 0, \alpha \neq 0$ and $\alpha \neq 1$ then we may use following equations:

$$e^{\pm x(n)} = 1 \pm x(n) + o(x(n)), \quad (6)$$
$$\text{and}$$
$$1 + \alpha x(n) + o(x(n)), \quad (7)$$
$$a + \alpha x(n) = a^\alpha + a^{\alpha-1} \alpha x(n) + o(x(n)) \quad (8)$$

Lemma 1. If

(i) $R^{(m_n)}_{k_n,s_n}(t)$ is a reliability function of the regular homogeneous series--"$m_n$ out of $k_n$" system given by (4),

(ii) $R^{(m)}(t)$ a non-degenerate reliability function given by (1),

(iii) $\lim_{n \to \infty} k_n = \infty, \lim_{n \to \infty} m_n = m = \text{const},$

$$\left(\frac{m_n}{k_n} \to 0 \text{ przy } k_n \to \infty\right),$$

(iv) $a_n > 0$, $b_n \in (-\infty, \infty)$ are some functions.

Then the assertion

$$\lim_{n \to \infty} R^{(m_n)}_{k_n,s_n}(a_n t + b_n) = R^{(m)}(t), \ t \in C_{R^{(m)}},$$

is equivalent to the assertion

$$\lim_{n \to \infty} k_n R^{(s)}_{k_n}(a_n t + b_n) = V(t), \ t \in C_V.$$ 

Proof. The proof may be found in [2] and [5].

Proposition 1. If components of the regular homogeneous regular series--"$m_n$ out of $k_n$" system have exponential reliability functions

$$R(t) = \begin{cases} 1 & \text{for } t \leq 0, \\ (t + 1)^{-c} & \text{for } t > 0, c > 0, \end{cases}$$

and moreover if

(i) $\lim_{n \to \infty} m_n = \text{const}, \lim_{n \to \infty} k_n = \infty, l_n = c$,

(ii) $a_n = k_n^{-\frac{c}{a}}, b_n = -1$,

then limit reliability functions of the system is

$$X(t) = \begin{cases} 1 & \text{for } t < 0, \\ \left(1 - \sum_{i=0}^{m-1} \exp(-t^{-\frac{1}{2}}) \frac{t^{-ic^2}}{i!}\right) & \text{for } t \geq 0. \end{cases}$$

Justification: According to Lemma 1 it is enough to show that

$$\lim_{n \to \infty} k_n R^{(s)}_{k_n}(a_n t + b_n) = V(t), \ t \in C_V,$$

where

$$V(t) = \begin{cases} \infty & t < 0, \\ t^{-\frac{c}{2}} & t > 0, \alpha > 0. \end{cases}$$

Since from (i) and (ii) we get for $n$ large enough

$$a_n t + b_n < 0 \text{ for } t \leq 0$$

and

$$a_n t + b_n > 0 \text{ for } t > 0.$$

Therefore

$$R(a_n t + b_n) = \begin{cases} 1 & \text{for } t \leq 0, \\ (a_n t + b_n + 1)^{-c} & \text{for } t > 0, c > 0. \end{cases}$$

According (i) and (ii) we get

$$k_n R^{(s)}_{k_n}(a_n t + b_n) = \begin{cases} k_n & \text{for } t \leq 0, \\ t^{-\frac{c}{2}} & \text{for } t > 0, c > 0, \end{cases}$$

what according (i) means, that (9) holds.

Proposition 2. If components of the regular homogeneous regular series--"$m_n$ out of $k_n$" system have Weibull reliability functions

$$R(t) = \begin{cases} 1 & \text{for } t < 0, \\ \exp(-\beta t^\alpha) & \text{for } t \geq 0, \alpha > 0, \beta > 0, \end{cases}$$

and moreover if

(i) $\lim_{n \to \infty} m_n = \text{const}, \lim_{n \to \infty} k_n = \infty, l_n = c$,

(ii) $a_n = k_n^{-\frac{c}{a}}, b_n = -1$,
Examples of series-“$m$ out of $k$” systems and their limit reliability functions

(i) $\lim_{n \to \infty} k_n = \infty$, $\lim_{n \to \infty} m_n = m = \text{const}$,

(ii) $a_n = \frac{b_n}{\alpha \log k_n}$, $b_n = \left( \frac{\log k_n}{\beta l_n} \right)^\frac{1}{\alpha}$,

then limit reliability functions of the system is

$$\mathcal{R}_3^{(m)}(t) = 1 - \sum_{i=0}^{m-1} \exp[-e^{-it}] \frac{e^{-it}}{l!} \quad \text{for } t \in (-\infty, \infty).$$

Justification: By Lemma 1 it is sufficient to show that

$$\lim_{n \to \infty} k_n R^l_n (a_n t + b_n) = e^{-t} \quad \text{for } t \in (-\infty, \infty).$$

According to (i) and (ii), for $n$ large enough, we get

$$a_n t + b_n = b_n \left( \frac{t}{\alpha \log k_n} + 1 \right) > 0 \quad \text{for } t \in (-\infty, \infty)$$

Hence for any $t \in (-\infty, \infty)$

$$R(a_n t + b_n) = \exp[-\beta(a_n t + b_n)^\alpha] = \exp[-\beta b_n \left( \frac{t}{\alpha \log k_n} + 1 \right)^\alpha] = \exp[- \log k_n \left( \frac{t}{\alpha \log k_n} + 1 \right)^\alpha].$$

Using (7) for $\left( \frac{t}{\alpha \log k_n} + 1 \right)^\alpha$ we get for $t \in (-\infty, \infty)$

$$R(a_n t + b_n) = \exp[- \log k_n - \frac{t}{l_n} - o(\frac{1}{l_n})].$$

Thus, for $t \in (-\infty, \infty)$

$$\lim_{n \to \infty} k_n R^l_n (a_n t + b_n) = \lim_{n \to \infty} k_n \exp[-\log k_n - t - o(1)] = \lim_{n \to \infty} \exp[-t - o(1)] = e^{-t}.$$

Directly, from the above Proposition, it follows next result.

Proposition 3. If components of the homogeneous regular series-“$m_n$ out of $k_n$” have exponential reliability functions:

$$R(t) = \begin{cases} 1 - \exp[-\beta t] & \text{for } t < 0, \\ \exp[-\beta t] & \text{for } t \geq 0, \beta > 0, \end{cases}$$

and moreover if

(i) $\lim_{n \to \infty} k_n = \infty$, $\lim_{n \to \infty} m_n = m = \text{const}$,

(ii) $a_n = \frac{1}{\beta l_n}$, $b_n = \frac{1}{\beta l_n} \log k_n$,

then limit reliability functions of the system is

$$\mathcal{R}_3^{(m)}(t) = 1 - \sum_{i=0}^{m-1} \exp[-\exp(-t)] \frac{e^{-it}}{l!}$$

for $t \in (-\infty, \infty)$.

Lemma 2. If

(i) $R^{(m)}_{k_n,t_n}(t)$ is given by (4) a reliability function of the homogeneous regular series--“$m_n$ out of $k_n$” system,

(ii) $\mathcal{R}(t)$ is non-degenerate reliability function,

(iii) $\lim_{n \to \infty} k_n = k$, $k > 0$, $\lim_{n \to \infty} m_n = m$, $0 < m \leq k$,

$$\lim_{n \to \infty} l_n = \infty,$$

(iv) $a_n > 0$, $b_n \in (-\infty, \infty)$ are some sequences of constants,

then the assertion

$$\lim_{n \to \infty} R^{(m)}_{k_n,t_n}(a_n t + b_n) = \mathcal{R}(t), \quad t \in C_{\mathcal{R}},$$

is equivalent to the assertion

$$\lim_{n \to \infty} R^{(l)}_{k_n}(a_n t + b_n) = \mathcal{R}_0(t), \quad t \in C_{\mathcal{R}_0},$$

where $\mathcal{R}_0$ in non-degenerate reliability function. Moreover for $t \in (-\infty, \infty)$,

$$\mathcal{R}(t) = 1 - \sum_{i=0}^{m-1} \left( \frac{k}{\mathcal{R}_0(t)} \right)^i \left[1 - \mathcal{R}_0(t)\right]^{k-i}.$$
Proof: The proof may be found in [5].

Proposition 4. If components of the regular homogeneous regular series-“\( m_n \) out of \( k_n \)” system have Weibull reliability functions

\[
R(t) = \begin{cases} 
1, & t < 0, \\
\exp[-\beta t^\alpha], & t \geq 0, \alpha > 0, \beta > 0,
\end{cases}
\]

moreover if \( \lim_{n \to \infty} m_n = m \) and pairs \((k_n, l_n)\) and \((a_n, b_n)\) fulfill conditions:

(i) \( \lim_{n \to \infty} k_n = k > 0 \), \( \lim_{n \to \infty} l_n = \infty \),

(ii) \( a_n = \left( \beta \frac{l_n}{k_n} \right)^\beta, b_n = 0 \),

then limit reliability functions of the system is

\[
\mathcal{R}_y^{(m)}(t) = \begin{cases} 
1 & \text{for } t < 0 \\
1 - \sum_{i=0}^{m-1} \left( \frac{k}{i} \right) \left[ \exp(-t^\alpha) \right]^i \left[ 1 - \exp(-t^\alpha) \right]^{k-i} & \text{for } t \geq 0, \alpha > 0.
\end{cases}
\]

From Proposition 4 the next result follows immediately.

Proposition 5. If components of the regular homogeneous regular series-“\( m_n \) out of \( k_n \)” have an exponential reliability functions

\[
R(t) = \begin{cases} 
1, & t < 0, \\
\exp[-\beta t], & t \geq 0, \beta > 0,
\end{cases}
\]

and moreover

\( \lim_{n \to \infty} m_n = m \)

and the pairs \((k_n, l_n)\) and \((a_n, b_n)\) fulfill the conditions:

(i) \( \lim_{n \to \infty} k_n = k > 0 \), \( \lim_{n \to \infty} l_n = \infty \),

(ii) \( a_n = \frac{1}{\beta l_n}, b_n = 0 \),

then limit reliability functions of the system is

\[
\mathcal{R}_y^{(m)}(t) = \begin{cases} 
1 & \text{for } t < 0, \\
1 - \sum_{i=0}^{m-1} \left( \frac{k}{i} \right) \left[ \exp(-t) \right]^i \left[ 1 - \exp(-t) \right]^{k-i} & \text{for } t \geq 0.
\end{cases}
\]

Lemma 3. If

(i) \( \overline{R}_{k_n,l_n}^{(m)}(t) \) is a reliability function of the homogeneous regular series-“\( m_n \) out of \( k_n \)” system given by (5),

(ii) \( \overline{\mathcal{R}}^{(m)}(t) \) is a non-degenerate reliability function given by (3),

(iii) \( n \to \infty, k_n \to \infty, \frac{m_n}{k_n} \to 1 \),

\( \overline{m_n} = k_n - m_n, \overline{m_n} \to \overline{m} = \text{const} \),

(iv) \( a_n > 0 \), \( b_n \in (-\infty, \infty) \) are some functions,

then the assertion

\[
\lim_{n \to \infty} \overline{R}_{k_n,l_n}^{(m)}(a_n t + b_n) = \overline{\mathcal{R}}^{(m)}(t), t \in C_{\overline{\mathcal{R}}^{(m)}},
\]

is equivalent to the assertion

\[
\lim_{n \to \infty} k_n l_n F(a_n t + b_n) = \overline{V}(t), t \in C_{\overline{V}}.
\]
Proof. The proof may be found in [5].

**Proposition 6.** If components of the regular homogeneous series—"m out of k"—system have Weibull reliability functions

\[ R(t) = \begin{cases} 
1 & \text{for } t < 0 \\
\exp[-\beta t^\alpha] & \text{for } t \geq 0, \alpha > 0, \beta > 0,
\end{cases} \]

and moreover if

(i) \( \lim_{n \to \infty} k_n = \infty \), \( \lim_{n \to \infty} (k_n - m_n) = \overline{m} = \text{const} \)

(ii) \( a_n = (\beta l_n k_n)\frac{\lambda}{\alpha} , b_n = 0 \),

then the limit reliability functions of the system is

\[ \overline{R}_2^{(\overline{m})}(t) = \begin{cases} 
1 & \text{for } t < 0, \\
\sum_{i=0}^{\overline{m}} \exp[-t^\alpha i] \frac{(t^\alpha)^i}{i!} & \text{for } t \geq 0, \alpha > 0.
\end{cases} \]

**Justification:** According to Lemma 3 it is enough to show that

\[ \lim_{n \to \infty} k_n l_n F(a_n t + b_n) = \overline{V}_2(t), t \in C_{\overline{V}_2}, \]

where

\[ \overline{V}_2(t) = \begin{cases} 
0 & \text{for } t < 0, \\
t^\alpha & \text{for } t > 0, \alpha > 0.
\end{cases} \]

According to (ii)

\[ a_n t + b_n = (\beta l_n k_n)\frac{\lambda}{\alpha} t. \]

Therefore

\[ a_n t + b_n < 0 \] for \( t < 0 \)

and

\[ a_n t + b_n \geq 0 \] for \( t \geq 0 \).

Because for \( t < 0 \)

\[ F(a_n t + b_n) = 1 - R(a_n t + b_n) = 0, \]

thus, for \( t < 0 \), we get

\[ \lim_{n \to \infty} k_n l_n F(a_n t + b_n) = 0. \]

However for any \( t \geq 0 \)

\[ F(a_n t + b_n) = 1 - R(a_n t + b_n) = 1 - \exp[-\beta(a_n t)^\alpha] \]

\[ = 1 - \exp[-\beta((\beta l_n k_n)^{-\alpha^{-1}} t)^\alpha] \]

\[ = 1 - \exp[(-l_n k_n)^{-1} t^\alpha]. \]

From the above for any \( t \geq 0 \), we get

\[ \lim_{n \to \infty} k_n l_n F(a_n t + b_n) = \lim_{n \to \infty} k_n l_n (1 - \exp[-\frac{t^\alpha}{k_n l_n}]) \]

Using (i) and (6), for \( t \geq 0 \), the above equation we can write as

\[ \lim_{n \to \infty} k_n l_n F(a_n t + b_n) = \lim_{n \to \infty} k_n l_n (1 - 1 + \frac{t^\alpha}{k_n l_n} - o(\frac{t^\alpha}{k_n l_n})) = t^\alpha. \]

From the above results the next proposition follows obviously.

**Proposition 7.** If components of the regular homogeneous regular series—"m out of k"—have exponential reliability functions

\[ R(t) = \begin{cases} 
1 & \text{for } t < 0, \\
\exp[-\beta t] & \text{for } t \geq 0, \beta > 0,
\end{cases} \]

and moreover if

(i) \( \lim_{n \to \infty} k_n = \infty \), \( \lim_{n \to \infty} (k_n - m_n) = \overline{m} = \text{const} \)

(ii) \( a_n = (\beta l_n k_n)\frac{\lambda}{\alpha} , b_n = 0 \),

then limit reliability functions of the system is

\[ \overline{R}_2^{(\overline{m})}(t) = \begin{cases} 
1 & \text{for } t < 0, \\
\sum_{i=0}^{\overline{m}} \exp[-t^\alpha i] \frac{(t^\alpha)^i}{i!} & \text{for } t \geq 0.
\end{cases} \]

**Lemma 4.** If

(i) \( R_k^{(m_n)}(t) , t \in (-\infty, \infty) \), is a reliability function of the homogeneous regular series—"m out of k"—given by (4),

(ii) \( \overline{R}^{(\overline{m})}(t) \), is a non-degenerate reliability function given by (2),

(iii) \( n \to \infty, k_n \to \infty \),

\[ \frac{m_n}{k_n} = \lambda + o\left(\frac{1}{\sqrt{k_n}}\right), 0 < \lambda < 1, \]

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(iv) \( a_n > 0, b_n \in (-\infty, \infty) \) are some functions, 
then the assertion 
\[
\lim_{n \to \infty} R_{k_n a_n}^{(m_n)} (a_n t + b_n) = \tilde{R}^{(\lambda)} (t) \text{ for } t \in C_{\tilde{R}^{(\lambda)}},
\]
is equivalent to the assertion 
\[
\lim_{n \to \infty} \frac{\sqrt{k_n + 1}[R^k (a_n t + b_n) - \lambda]}{\sqrt{\lambda(1 - \lambda)}} = L(t)
\]
for \( t \in C_L \).

**Proof.** The proof may be found in [5]. \(\square\)

**Proposition 8.** If components of the regular homogeneous regular series—"\(m_n\) out of \(k_n\)"—system have Weibull reliability functions:

\[
R(t) = \begin{cases} 
1, & t < 0, \\
\exp[-\beta t^\alpha], & t \geq 0, \alpha > 0, \beta > 0,
\end{cases}
\]

and moreover

(i) \( \lim_{n \to \infty} k_n = \infty, \frac{m_n}{k_n} = \lambda + o \left( \frac{1}{\sqrt{k_n}} \right), \)
\[0 < \lambda < 1,\]

(ii) \( a_n = \frac{\sqrt{1 - \lambda}}{\alpha(\beta t_n^\alpha) \sqrt{\alpha k_n (-\log \lambda)^{\alpha-1}}} \),
\[b_n = -\frac{\log \lambda}{\beta t_n^\alpha},\]

then limit reliability functions of the system is

\[
\tilde{R}^{(\lambda)} (t) = 1 - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{x^2}{2}} dx \quad \text{for } t \in (-\infty, \infty).\]

**Justification:** According to Lemma 4 it is enough to show that for \( t \in (-\infty, \infty) \)

\[
\lim_{n \to \infty} \frac{\sqrt{k_n + 1}[R^k (a_n t + b_n) - \lambda]}{\sqrt{\lambda(1 - \lambda)}} = -t.
\]

From (i) and (ii), because for \( n \) large enough and for \( t \in (-\infty, \infty) \), we have

\[
a_n t + b_n =
\]

\[
\lim_{n \to \infty} R_{k_n a_n}^{(m_n)} (a_n t + b_n) = \tilde{R}^{(\lambda)} (t) \text{ for } t \in C_{\tilde{R}^{(\lambda)}},
\]

\[
L(t) = \frac{\sqrt{k_n + 1}[R^k (a_n t + b_n) - \lambda]}{\sqrt{\lambda(1 - \lambda)}}
\]

it follows that

\[
R(a_n t + b_n) = \exp[-\beta \left( \frac{1}{\alpha(\beta t_n^\alpha) \sqrt{\alpha k_n (-\log \lambda)^{\alpha-1}}} \right) t + \left( -\log \lambda \right)^\frac{1}{\alpha}] \]

and next for large \( n \) and \( t \in (-\infty, \infty) \) we get

\[
\lim_{n \to \infty} \frac{\sqrt{k_n + 1}[R^k (a_n t + b_n) - \lambda]}{\sqrt{\lambda(1 - \lambda)}} = \lim_{n \to \infty} \frac{\sqrt{k_n + 1}[\exp[-\left( \frac{1}{\alpha(\beta t_n^\alpha) \sqrt{\alpha k_n (-\log \lambda)^{\alpha-1}}} \right) t + \left( -\log \lambda \right)^\frac{1}{\alpha}] - \lambda]}{\sqrt{\lambda(1 - \lambda)}}
\]

Using (8) we get

\[
\left( \frac{\sqrt{1 - \lambda}}{\alpha(\beta t_n^\alpha) \sqrt{\alpha k_n (-\log \lambda)^{\alpha-1}}} \right) t + \left( -\log \lambda \right)^\frac{1}{\alpha} = \frac{1}{\alpha(\beta t_n^\alpha) \sqrt{\alpha k_n (-\log \lambda)^{\alpha-1}}} \left( \frac{\sqrt{1 - \lambda}}{\alpha(\beta t_n^\alpha) \sqrt{\alpha k_n (-\log \lambda)^{\alpha-1}}} \right) t + \left( -\log \lambda \right)^\frac{1}{\alpha} + \alpha(-\log \lambda)^\frac{1}{\alpha} t + o \left( \frac{1}{\sqrt{k_n}} \right) = -\log \lambda + \sqrt{1 - \lambda} + o \left( \frac{1}{\sqrt{k_n}} \right)
\]

Continuing transformations (11) we get

\[
\sqrt{k_n + 1}[\exp[-\left( \frac{1}{\alpha(\beta t_n^\alpha) \sqrt{\alpha k_n (-\log \lambda)^{\alpha-1}}} \right) t + \left( -\log \lambda \right)^\frac{1}{\alpha}] - \lambda] \frac{1}{\alpha(\beta t_n^\alpha) \sqrt{\alpha k_n (-\log \lambda)^{\alpha-1}}} t + o \left( \frac{1}{\sqrt{k_n}} \right)
\]

From the above, using (6), we get for \( t \in (-\infty, \infty) \)
Examples of series-“m out of k” systems and their limit reliability functions

$$\lim_{n \to \infty} \Phi^{(\lambda)}(t) = 1 - \frac{1}{2\pi} \int_{-\infty}^{\infty} x^2 e^{-\frac{x^2}{2}} dx$$ for $t \in (-\infty, \infty)$.

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**References**


